Average-Case Robustness of Stable Matchings: (Counting) Complexity and Experiments

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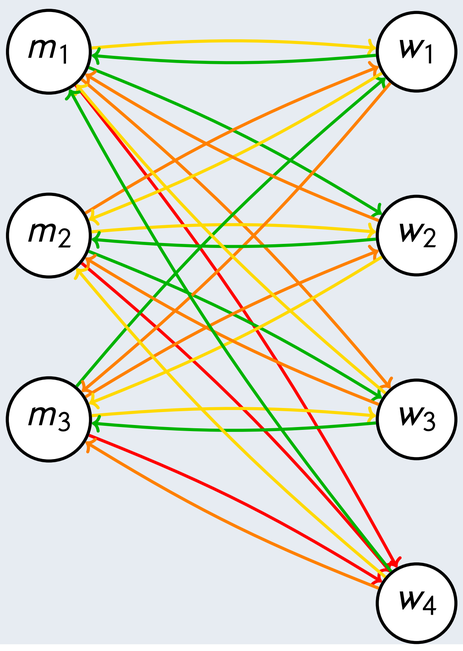
Stable Matchings, Pairs and Agents



Stable Marriage: $\mathcal{I} = (U, W, \mathcal{P})$

- **Blocking pair:** (m, w) s.t. $m \succ_w M(w)$ and $w \succ_m M(m)$

1. **Stable Matching:** no pair is blocking
2. **Stable Pair:** there is a stable matching including this pair
3. **Stable Agent:** there is a stable matching where this agent is assigned

On ordering $\bullet \succ \bullet \succ \bullet \succ \bullet$:

- Stable matching $\{(m_i, w_i) \mid i \in \{1, 2, 3\}\}$
- Stable pair (m_1, w_2)
- Unstable agent w_4

Robustness Measures and Computational Problems

- We consider two distance measures between preference profiles:
  - **Swap distance:** min number of swaps of adjacent agents in preferences
  - **Deletion distance:** min number of deletions of agents
- **Worst-Case Robustness of stable matching** M**:** Smallest number of changes to make M unstable
- **Average-Case Robustness of stable matching** M**:** *Probability* that M is stable after a certain number of changes have been made (uniformly at random)

Matching/Pair/Agent - Swap/Deletion - Robustness
Input: A SM instance $\mathcal{I}$, a budget $\ell \in \mathbb{N}$, and a matching M/pair p/agent a
Output: Is there a SM instance $\mathcal{I}'$ at swap/deletion distance at most ℓ from $\mathcal{I}$ such that $M/\{u, w\}/a$ is not stable in $\mathcal{I}'$?

- The corresponding counting problem asks for the **number** of preference profiles at swap/deletion distance ℓ such that $M/p/a$ is not stable
- Note: counting all profiles at some distance is poly-time

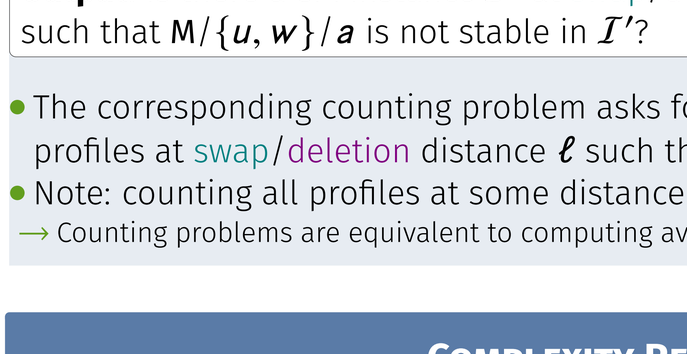
→ Counting problems are equivalent to computing average-case robustness

Complexity Results

	Matching		Matching (bps)		Pair		Agent	
	decision	counting	decision	counting	decision	counting	decision	counting
Swap	P	#P-h. (FPRAS)	NP-h.	–	NP-h.	–	NP-h.	–
Delete	P	#P-h. [1]	NP-h.	–	?	#P-h.	P	#P-h.

Counting complexity

#Matching Swap Robustness is #P-hard



- From #Bipartite 2-Sat with no Negations (#P-hard [2])
- M contains one pair for each literal (green edges)
- (m_p, w_q) form a blocking pair for each $p \vee q$ (red edges)
- Preferences of m_p (if $p = 1$) or $w_{\bar{p}}$ (if $p = 0$) are modified
- Doubling of the construction to ensure bijection

#Agent Delete Robustness is #P-hard

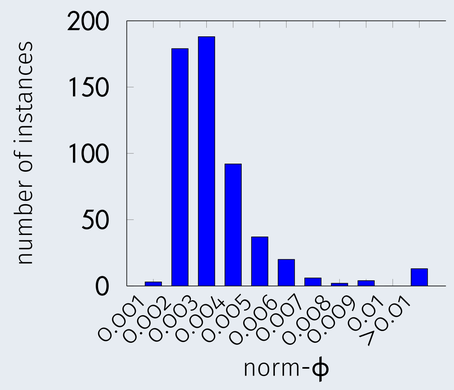
- From #Edge Cover (#P-hard [3])
- Associate solutions to edge sets E′ whose removal isolate at least one vertex
- The remaining set is a non-edge-cover
- Delete one agent for each edge in E′ (detailed construction is technical and is omitted here)

An FPRAS for Matching-Swap-Robustness

1. **Polynomial-factor approximation**:
   - Idea: For all pairs, count preference profiles where at least this pair is blocking, and sum everything together (this will overcount profiles with multiple blocking pairs)
   - For every pair (m, w) and every preference list, count # permutations at swap distance $\leq \ell$ where $M(m) \succ w$
   - Deduce the number of profiles where (m, w) is blocking
   - Repeating for all $n^2 - n$ blocking pairs will give an overcounting of at most $n^2 - n$
2. **Estimate the factor of overcounting**:
   - Idea: Determine the average number of times we count each profile by sampling
   - Fix an ordering of the pairs
   - In iteration i, sample a pair p and a profile where p is blocking, set $X_i = 1$ if p is the *first* blocking pair, $X_i = 0$ else
   - Count the percentage of samples where p is the first blocking pair in the profile
   - For b the first approximation and $s = O((\frac{n}{\epsilon})^2)$ the number of iterations, output

$$\frac{b}{s} \cdot \sum_{i=1}^{s} X_i$$

Experimental Robustness of Matchings



- Instances are drawn from different statistical cultures (*Impartial, Euclidean,...*)
- Worst-case robustness of *almost all* instances is 1

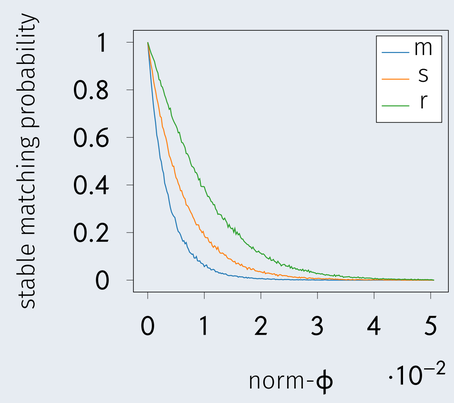
→ study average-case!

- Negative complexity results

→ model noise by the *Mallows model* [4]

- Most instances exhibit very low average-case robustness, but not all
- Uniformly-at-random instances are less robust than more organized instances

Comparing different methods to compute stable matchings



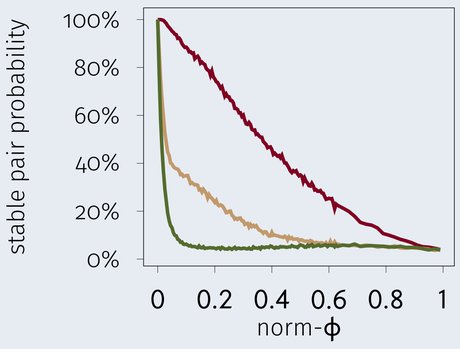
- **Robust stable matching heuristic:** stable matching that maximizes proximity of pairs to being blocking:

$$\sum_{k=1}^{d} n^{d-k} \cdot \#\beta(M, k)$$

- We compare:
  1. Men-optimal stable matching
  2. Summed-rank minimizing stable matching [5]
  3. Robust stable matching

→ Men-optimal matchings are less robust!

Experimental Robustness of Pairs



- Pairs have much higher robustness than matchings
- Robustness curves are not always monotone
- Pairs in one instance can have largely different robustness
- On the left: Three different stable pairs of one instance

Future work

- Complexity of Pair Deletion Robustness
- Circumventing hardness by additional parametrized and approximability results
- Experiments on real-world data sets

Full arXiv-Version:

[1] Niclas Boehmer, Robert Bredereck, Klaus Heeger, and Rolf Niedermeier. Bribery and control in stable marriage. *J. Artif. Intell. Res.*, 71:993–1048, 2021.

[2] J Scott Provan and Michael O Ball. The complexity of counting cuts and of computing the probability that a graph is connected. *SIAM J. Comput.*, 12(4):777–788, 1983.

[3] Russ Bubley and Martin E. Dyer. Graph orientations with no sink and an approximation for a hard case of #sat. In *Proceedings of the Eighth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA '97)*.

[4] C. Mallows. Non-null ranking models. *Biometrica*, 44:114–130, 1957.

[5] Donald Ervin Knuth. Marriages stables. *Technical report*, 1976.